An Asymptotic Numerical Method for Inverse Elastic Shape Design

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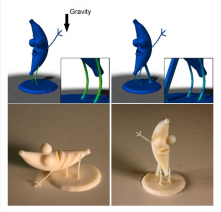
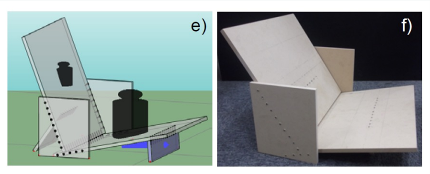
Fabrication-aware design



[Stava et al. 2012; etc.]

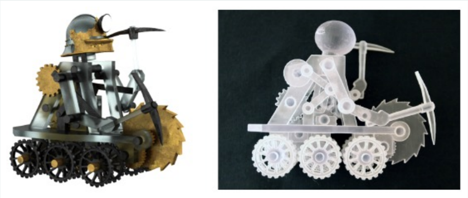


[Umetani et al. 2012; etc.]

[Wang et al. 2013]

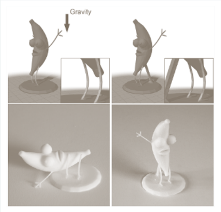
[Bacher et al. 2012; etc.]

[Prevost et al. 2013]



[Coros et al. 2013; etc.]

Fabrication-aware design



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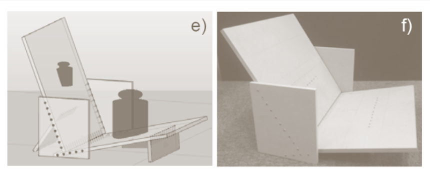


[Umetani et al. 2012; etc.]

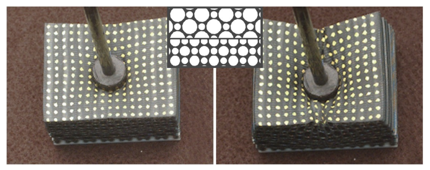
[Wang et al. 2013]

Our focus: Elastic Fabrication

[Bacher et al. 2012; etc.]

[Prevost et al. 2013]

[Coros et al. 2013; etc.]



[Bickel et al. 2010]

[Skouras et al. 2013]

Introduction

Design target

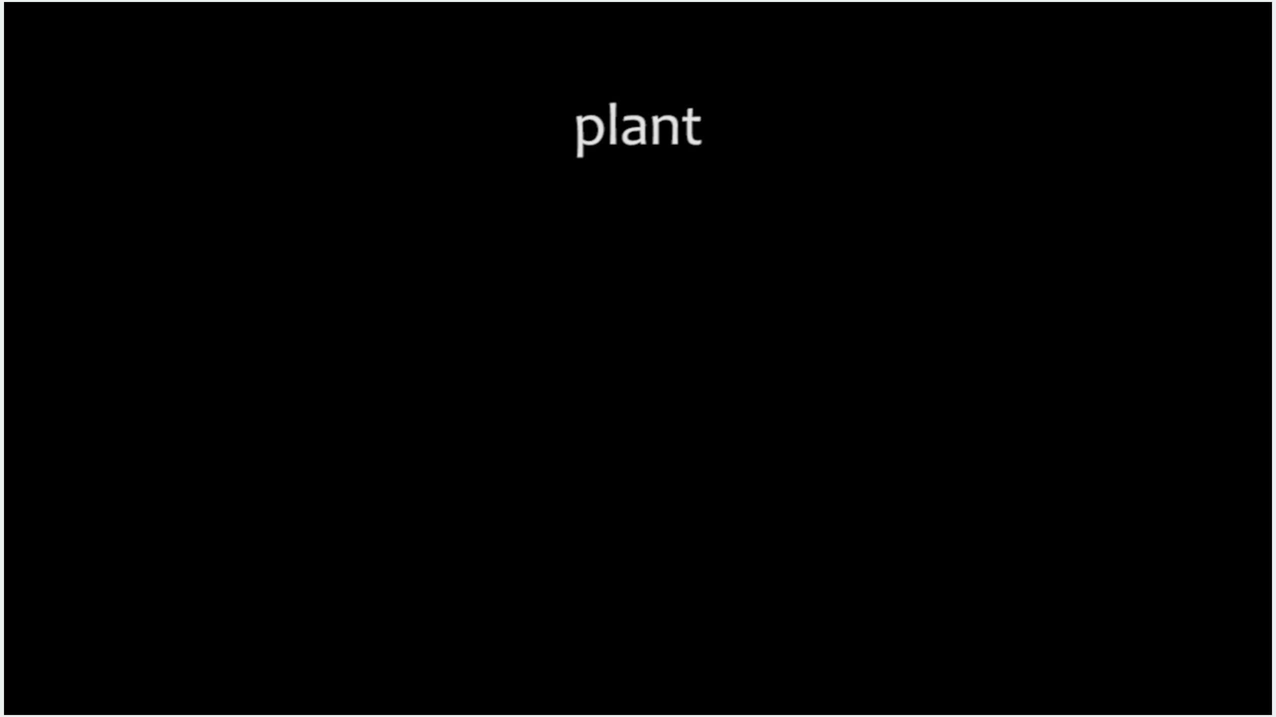
fabrication

Inverse computation

Rest shape

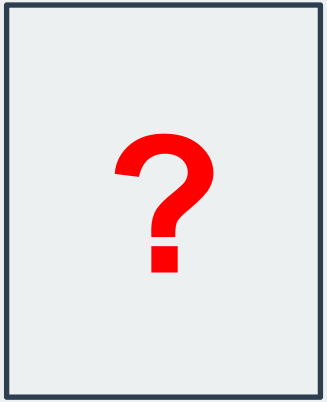
fabrication

Introduction



Mathematical definition

Rest shape: $\boldsymbol{X}$



Inverse static equilibrium

$$f(x, X) + g = 0$$

Target shape: $\boldsymbol{x}$

Inverse static equilibrium

internal   external

$$\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{X}) + \boldsymbol{g} = 0$$

Hyperelastic neo-Hookean model:
strain energy density function

$$W(x, X) = \left(\frac{\mu}{2}\left(J^{-\frac{2}{3}} I_c - 3\right) + \frac{\kappa}{2}(J-1)^2\right)$$

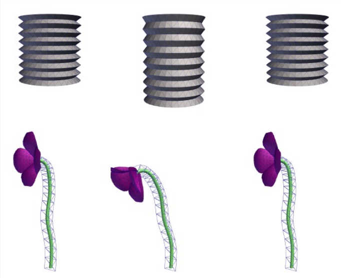
$J = \det(F)$ $I_c = \mathrm{Tr}(F^T F)$

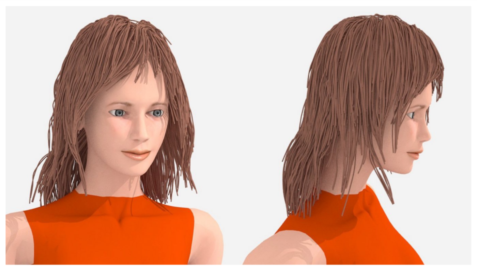
Highly nonlinear
High dimensional

Predictive for fabrication
Used in [Skouras et al. 2013]…

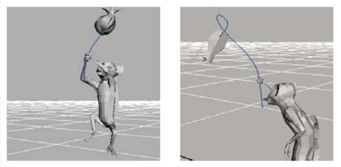
Previous work on inverse problem

- Spring-mass system
  - [Twigg and Kacic-Alesic 2011]

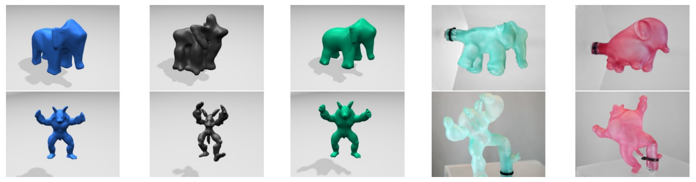




- Rod-like geometries
  - [Hadap 2006]
  - [Derouet-Jourdan et al. 2010, 2012]



- Computational design and fabrication
  - [Bickel et al. 2012]
  - [Skouras et al. 2012]



Difficulties of Newton-type Methods

$$\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{X}) + \boldsymbol{g} = \mathbf{0}$$

- Relies on a good initial guess
- May converge very slowly or even not converge

Asymptotic numerical method (ANM)

Asymptotic numerical method (ANM)

$$\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{X}) + \boldsymbol{g} = 0$$

Main idea of ANM

$$f(x, X) + g = 0$$

Main idea of ANM

Parameterized equation:

$$\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{X}) + \boldsymbol{\lambda g} = \boldsymbol{0}$$

Main idea of ANM

$$f\,(x, X) + \lambda g = 0$$

Main idea of ANM

?

$f\,(x, X) + \lambda g = 0$

$(X, \lambda = 1)$

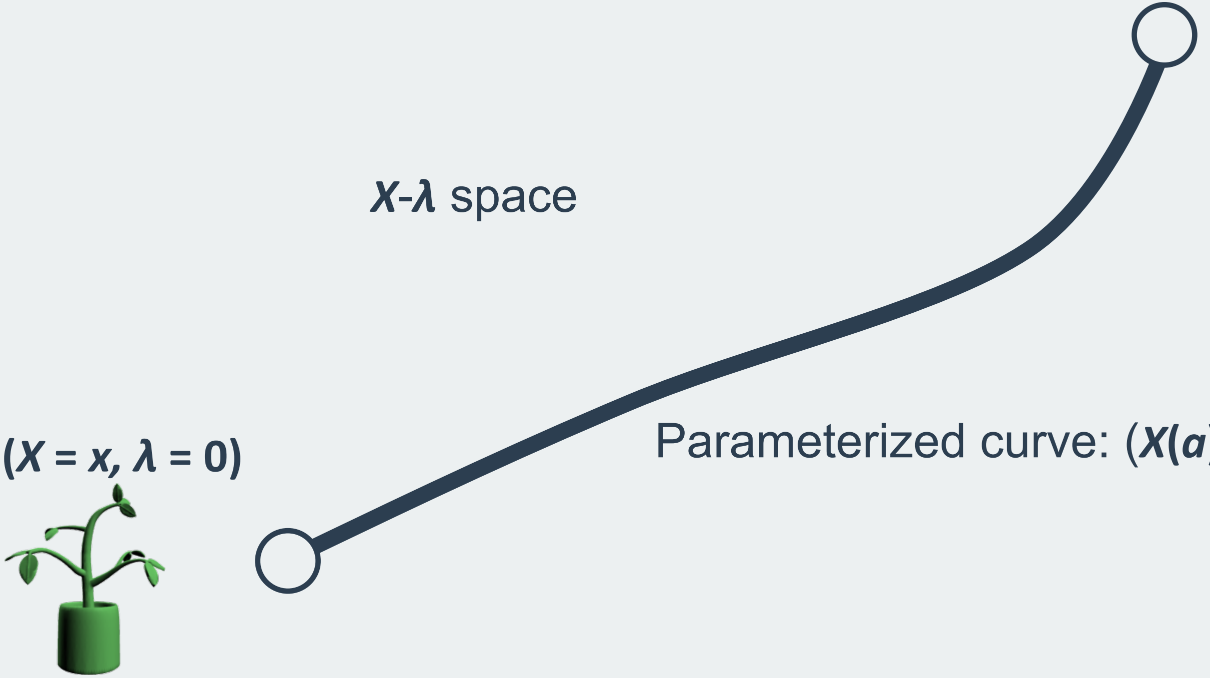
(X, λ)

$(X = x, \lambda = 0)$

Main idea of ANM

$f\,(x, X) + \lambda g = 0$

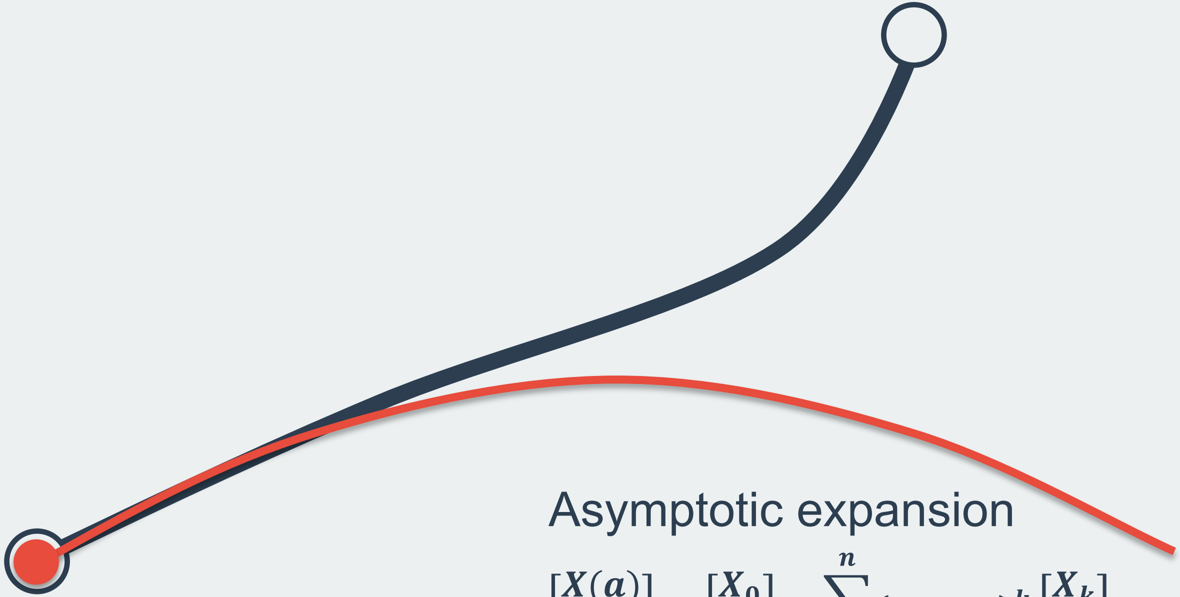
$(X, \lambda = 1)$



Main idea of ANM

$f\,(x, X) + \lambda g = 0$

$(X, \lambda = 1)$



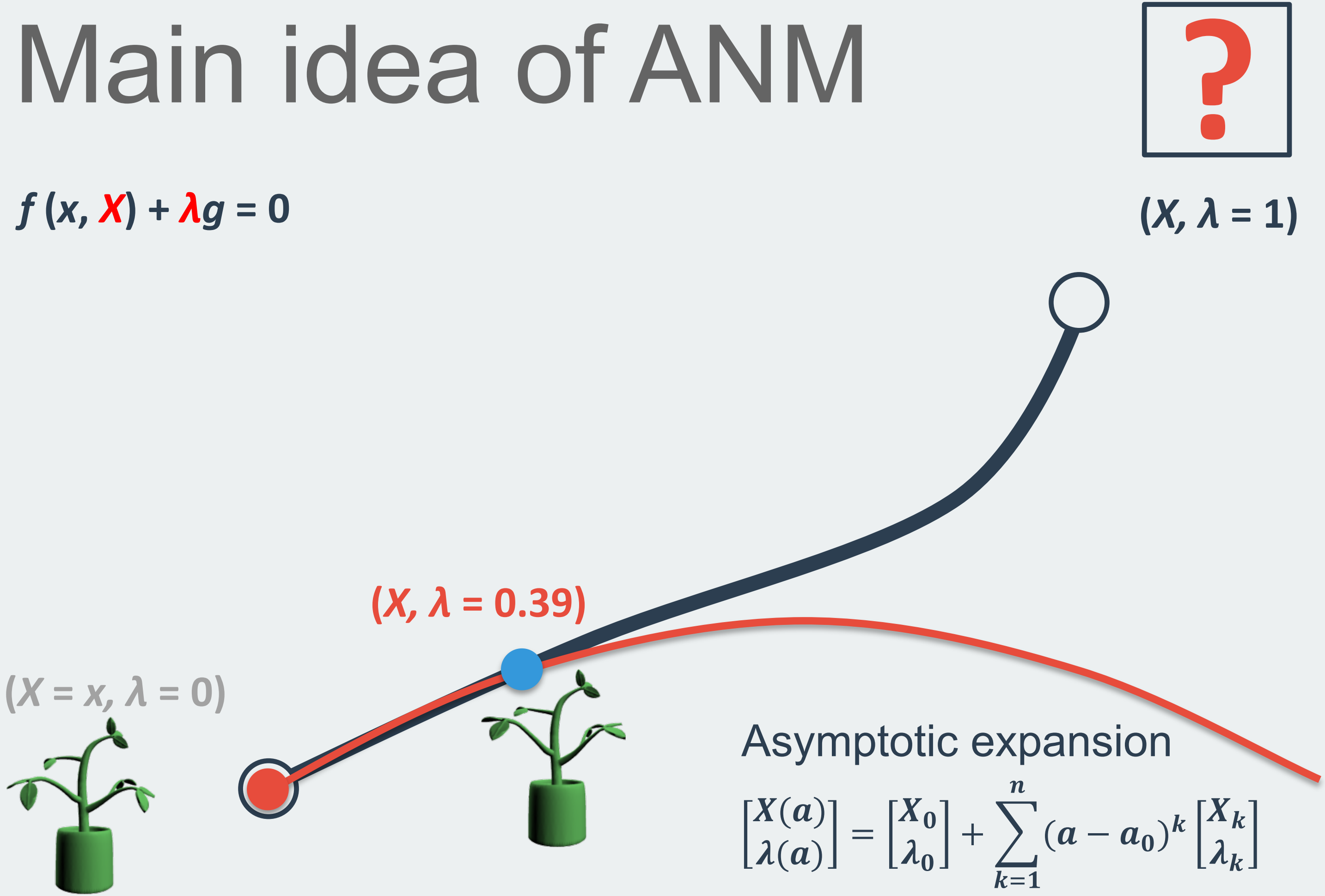
$(X = x, \lambda = 0)$

Asymptotic expansion

$$\begin{bmatrix} X(a) \\ \lambda(a) \end{bmatrix} = \begin{bmatrix} X_0 \\ \lambda_0 \end{bmatrix} + \sum_{k=1}^{n} (a - a_0)^k \begin{bmatrix} X_k \\ \lambda_k \end{bmatrix}$$

Main idea of ANM

$f\,(x, X) + \lambda g = 0$



Asymptotic expansion

$$\begin{bmatrix} X(a) \\ \lambda(a) \end{bmatrix} = \begin{bmatrix} X_0 \\ \lambda_0 \end{bmatrix} + \sum_{k=1}^{n} (a - a_0)^k \begin{bmatrix} X_k \\ \lambda_k \end{bmatrix}$$

Main idea of ANM

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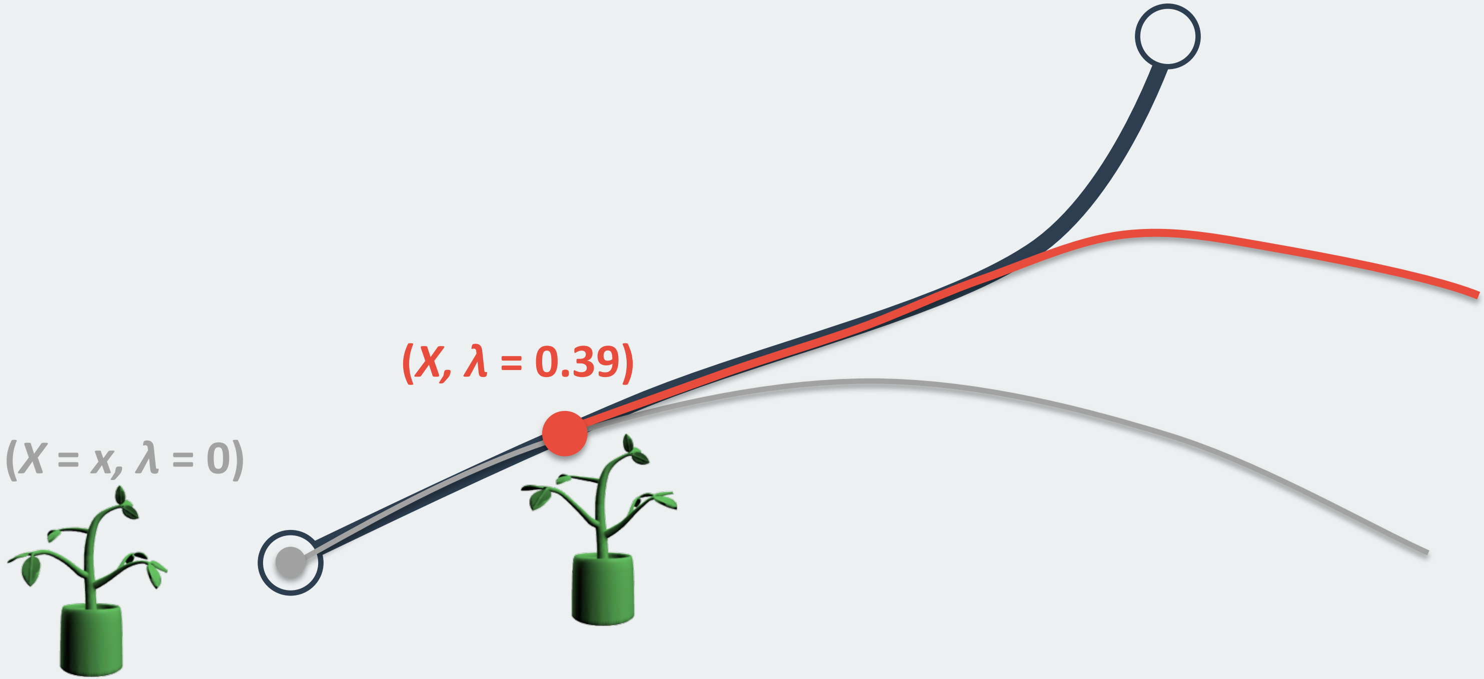
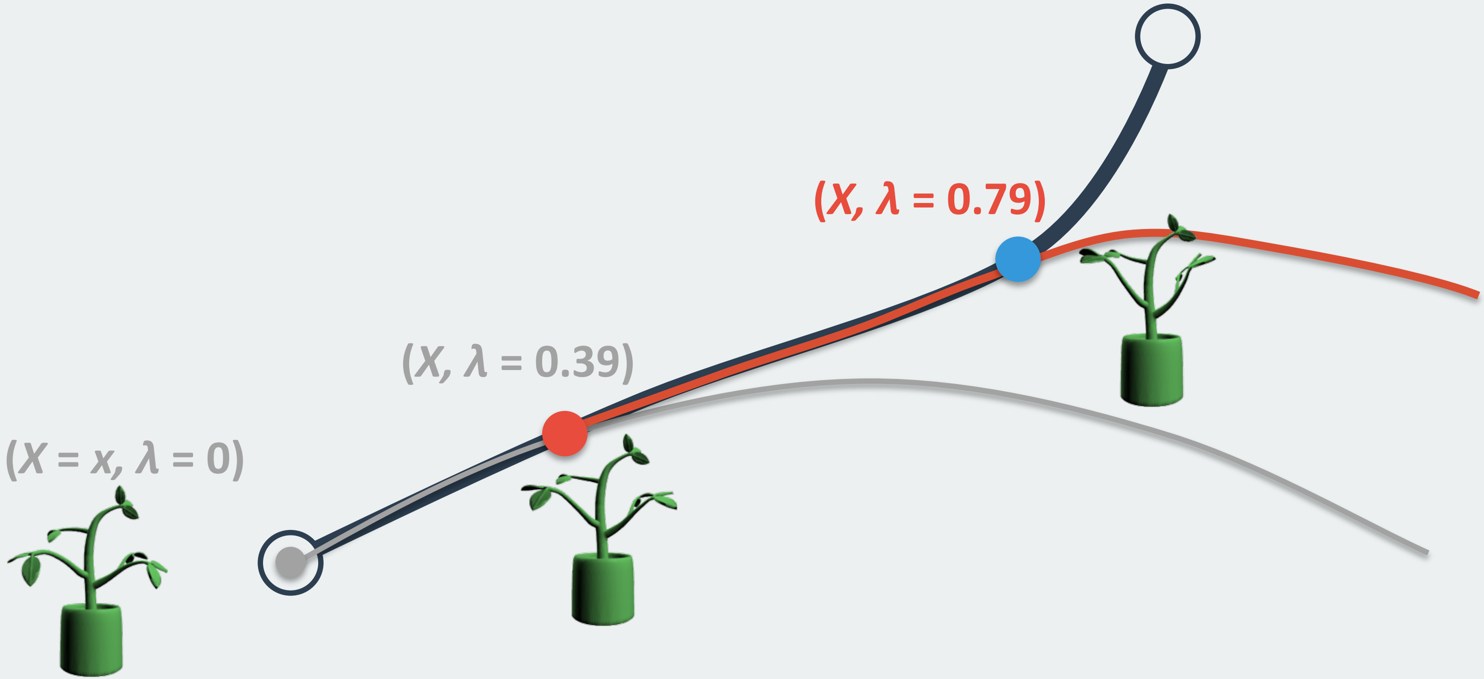


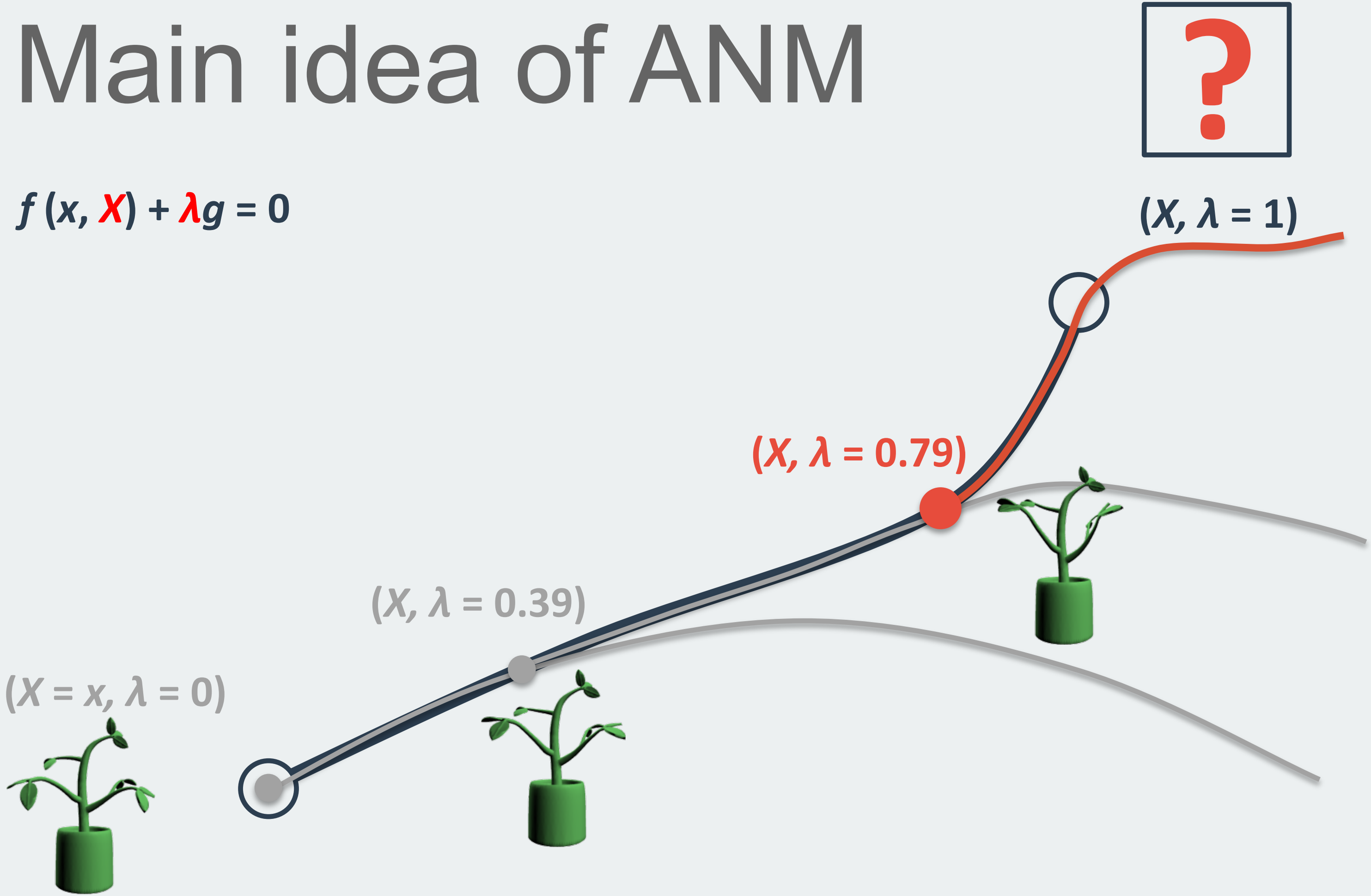
Main idea of ANM

$f\,(x, X) + \lambda g = 0$



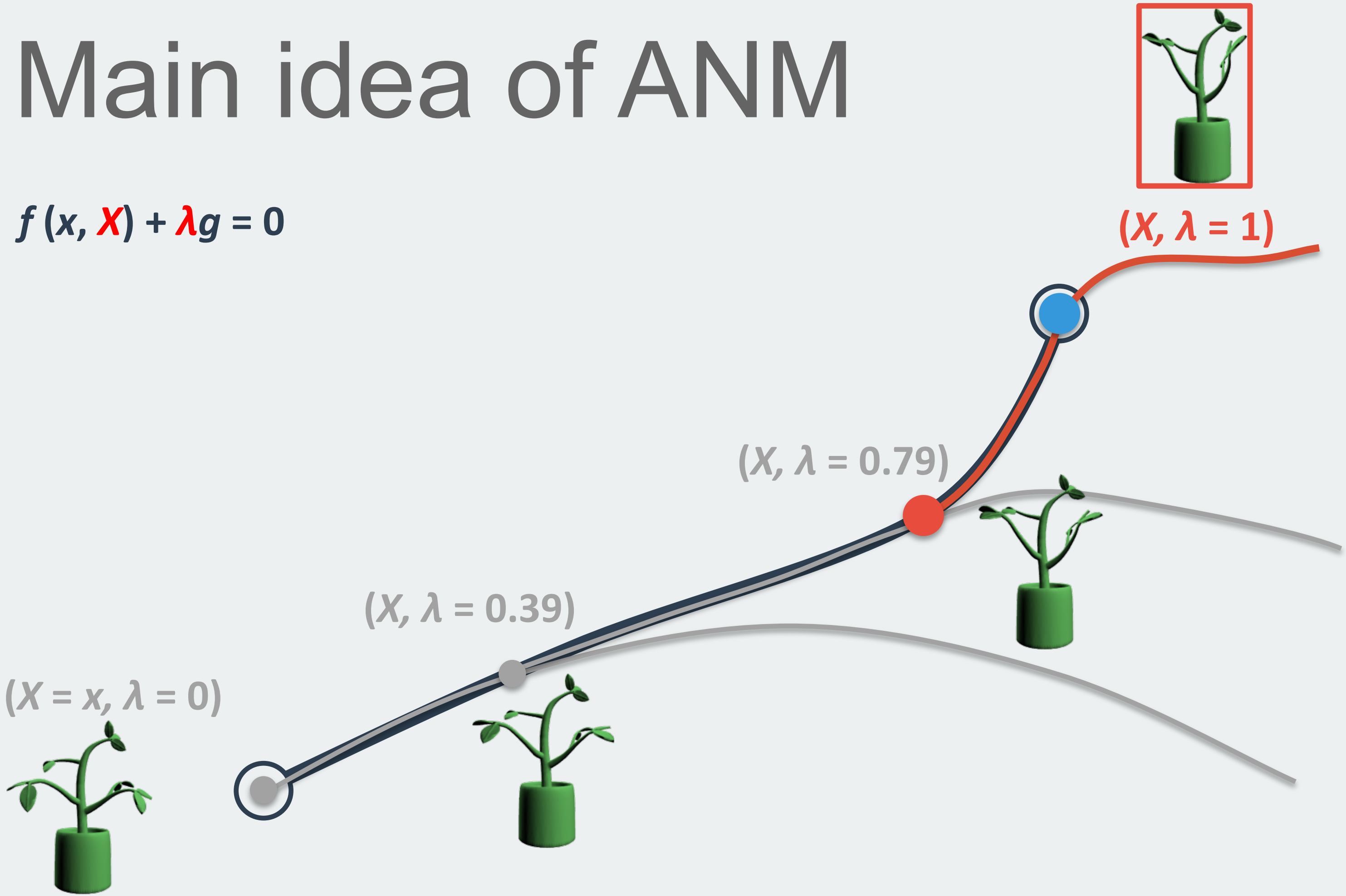
Main idea of ANM

$f(x, X) + \lambda g = 0$



Main idea of ANM

$f(x, X) + \lambda g = 0$



Mathematical details of ANM

- Asymptotic expansion

coefficients

$$\begin{bmatrix} X(a) \\ \lambda(a) \end{bmatrix} \approx \begin{bmatrix} X_0 \\ \lambda_0 \end{bmatrix} + \sum_{k=1}^{n} (a - a_0)^k \begin{bmatrix} X_k \\ \lambda_k \end{bmatrix}$$

Mathematical details of ANM

- Quadratic equation

linear bilinear

$$f(x, X) = L_0 + \boxed{L}[X] + \boxed{Q}[X, X]$$

Mathematical details of ANM

- Quadratic equation

$$f(x, X) + \lambda g = L_0 + L[X] + Q[X, X] + \lambda g = 0$$

$$\begin{bmatrix} X(a) \\ \lambda(a) \end{bmatrix} \approx \begin{bmatrix} X_0 \\ \lambda_0 \end{bmatrix} + \sum_{k=1}^{n} (a - a_0)^k \begin{bmatrix} X_k \\ \lambda_k \end{bmatrix}$$

Mathematical details of ANM

$$f\big(x, X(a)\big) + \lambda(a) g = L_0 + L[X_0] + Q[X_0, X_0] + (a - a_0)(L[X_1] + 2Q[X_0, X_1])$$
$$+ \sum_{k=2}^{n} (a - a_0)^k \left(L[X_k] + 2Q[X_0, X_k] + \sum_{t=1}^{k-1} Q[X_t, X_{k-t}] \right)$$
$$+ \left[\lambda_0 + (a - a_0)\lambda_1 + \sum_{k=2}^{n} (a - a_0)^k \lambda_k \right] g$$
$$= 0$$

Mathematical details of ANM

$$f(x, X(a)) + \lambda(a) g = L_0 + L[X_0] + Q[X_0, X_0] + (a - a_0)(L[X_1] + 2Q[X_0, X_1])$$
$$+ \sum_{k=2}^{n} (a - a_0)^k \left(L[X_k] + 2Q[X_0, X_k] + \sum_{t=1}^{k-1} Q[X_t, X_{k-t}] \right)$$
$$+ \left[\lambda_0 + (a - a_0)\lambda_1 + \sum_{k=2}^{n} (a - a_0)^k \lambda_k \right] g$$
$$= 0$$

order 0

order 1

order k

Mathematical details of ANM

1st order: $L[X_1] + 2Q[X_0, X_1] + \lambda_1 g = 0$

higher order:

$$L[X_k] + 2Q[X_0, X_k] + \sum_{t=1}^{k-1} Q[X_t, X_{k-t}] + \lambda_k g = 0$$

$$X_k^T X_1 + \lambda_k \lambda_1 = \delta_{k1}, k = 1 \ldots n$$

Unique

$$A \begin{bmatrix} X_k \\ \lambda_k \end{bmatrix} = b_k$$

Mathematical details of ANM

- Quadratic assumption

$$f(x, X) = L_0 + L[X] + Q[X, X]$$

- Highly nonlinear internal force?

Mathematical details of ANM

- A simple example: $f(x) = x^{3/2}$

- Augment the function

$$\tilde{f}(x,y) = \begin{bmatrix} f(x,y) \\ x \end{bmatrix} = \begin{bmatrix} xy \\ y^2 \end{bmatrix}$$

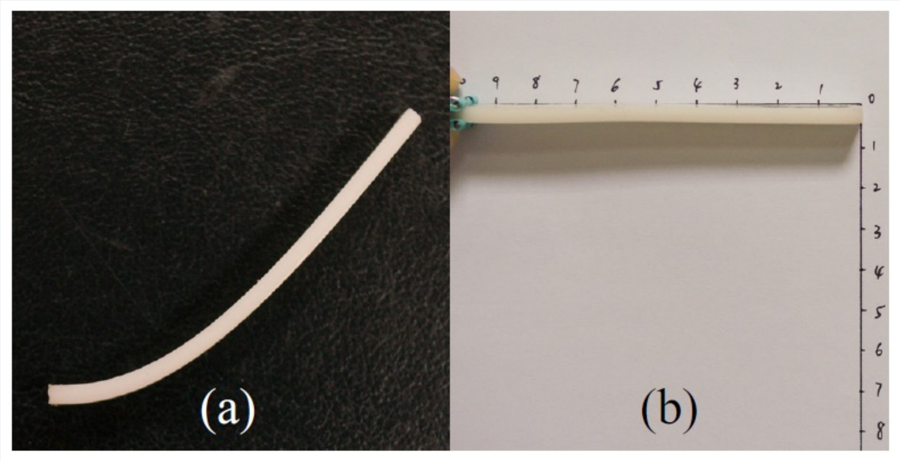
Mathematical details of ANM

$$f_i = \sum_{t \in \mathrm{adj}(x_i)} \sigma^t n_i^t$$

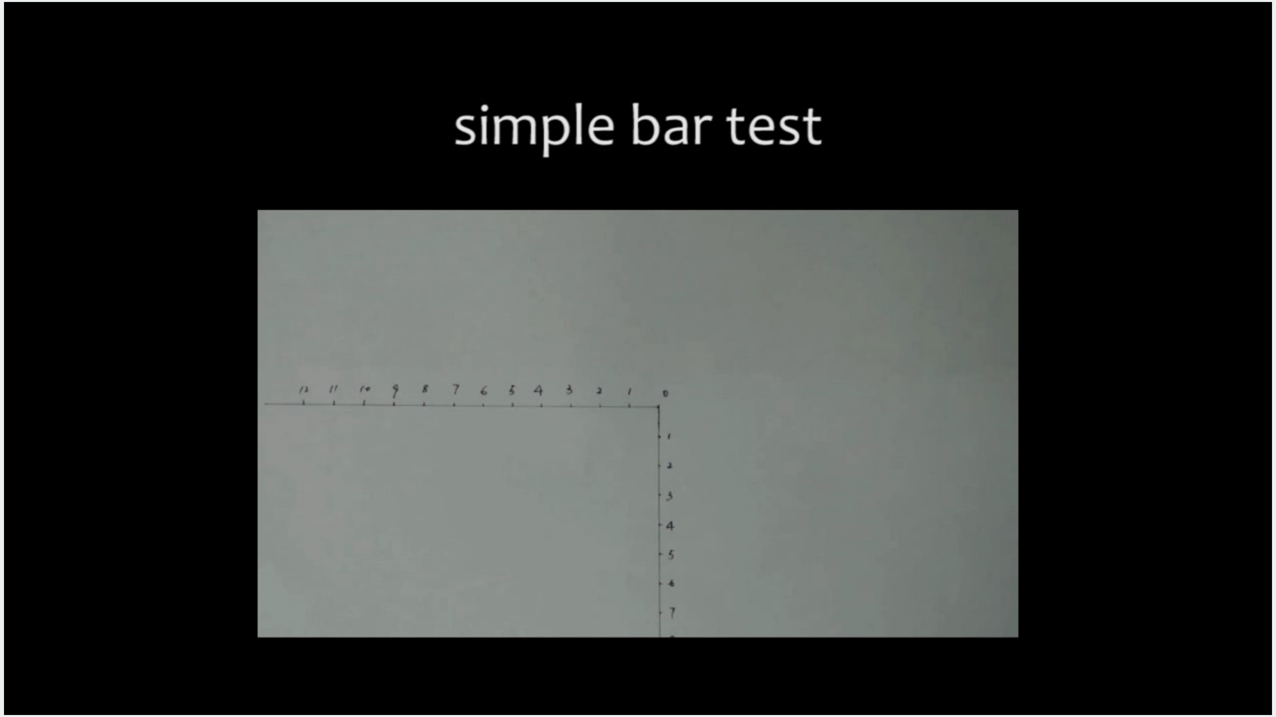
$$\begin{cases} \sigma = \mu J^{-\frac{5}{3}} b - \frac{\mu}{3} J^{-\frac{5}{3}} I_c I + \kappa(J-1) I \\ s^{(5)} = s^{(2)} J^{inv} \\ s^{(2)} = s^{(1)} s^{(1)} \\ J^{inv} = s^{(2)} s^{(1)} \\ J J^{inv} = 1 \\ J = \epsilon_{lmn} \tilde{F}_{lm} F_{n3} \\ \tilde{F}_{lm} = F_{l1} F_{m2} \\ I_c = I : b \\ b = F F^T \\ F = [x][\mathrm{X}]^{inv} \\ [X][X]^{inv} = I \end{cases}$$

Fabrication Results

- Simple bar

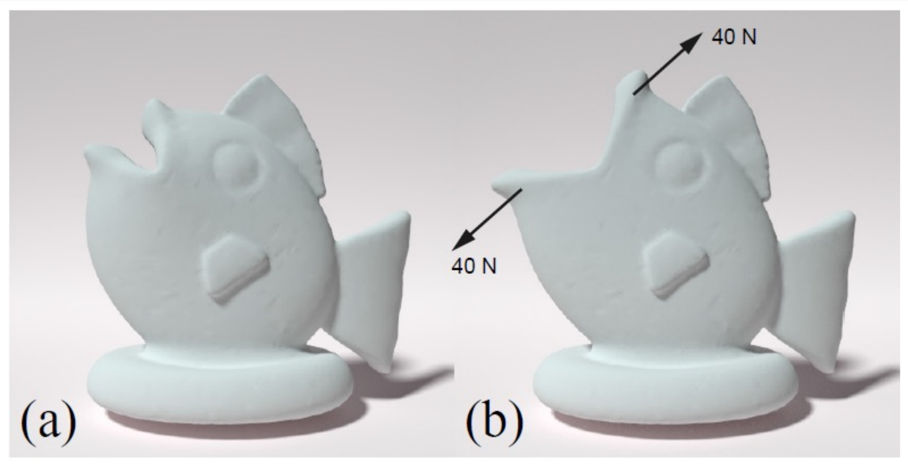


Fabrication Results

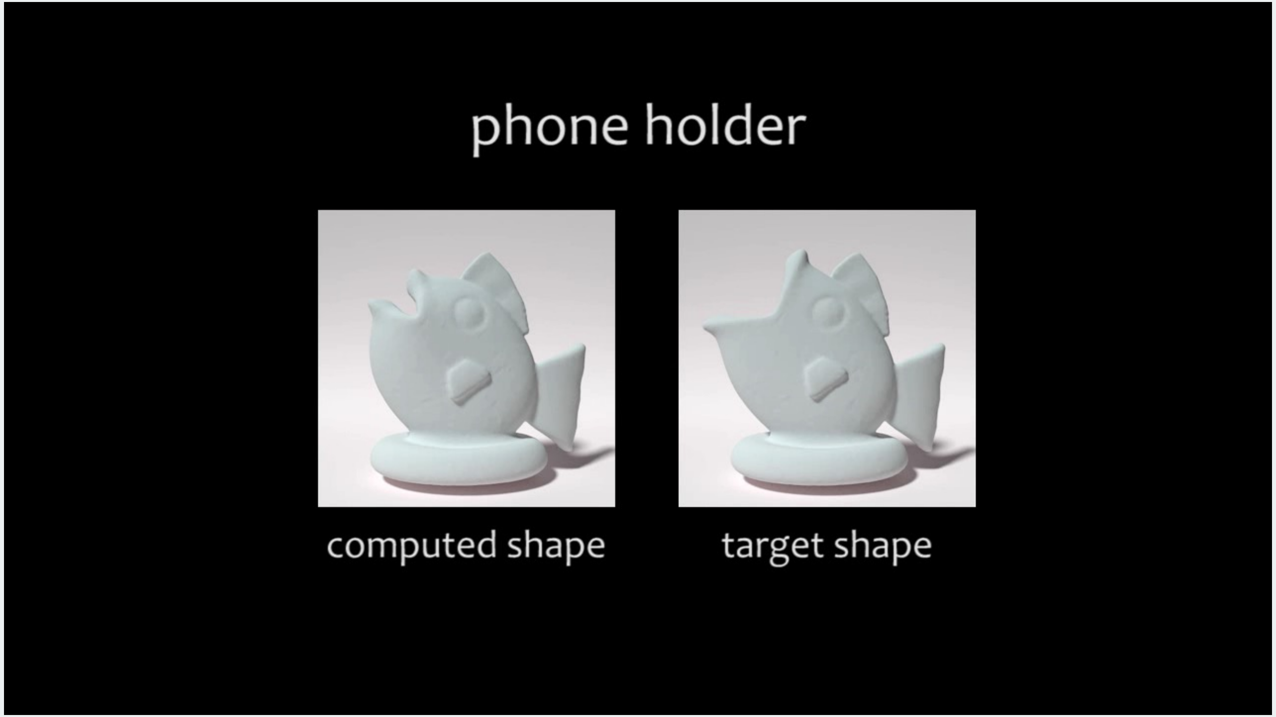


Fabrication Results

- Phone holder



Fabrication Results

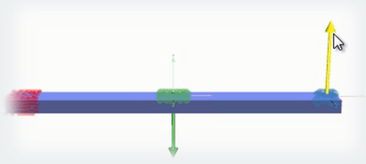


Running time

Model	LevMar solver		ANM solver		Speedup
	Time	#Iter.	Time	#Iter.	
bar	25m18s	5402	2.38s	2	638×
	8m23s	1750			211×
plant	1h2m34s	7594	7.07s	3	531×
	28m13s	3455			239×
holder	4m25s	95	12.82s	3	21×
	9m20s	216			43×
hanger	16m22s	629	26.83s	1/3	37×
	17m21s	675			38×
eagle	1h50m2s	4691	4.49s	1	1470×
	40m24s	780			539×
dinosaur	22m51s	993	28.17s	1/2/4/2	49×
	23m39s	1036			50×
bifur3	18m15s	1996	5.22s	3	210×
	14m53s	1613			171×

Extensions

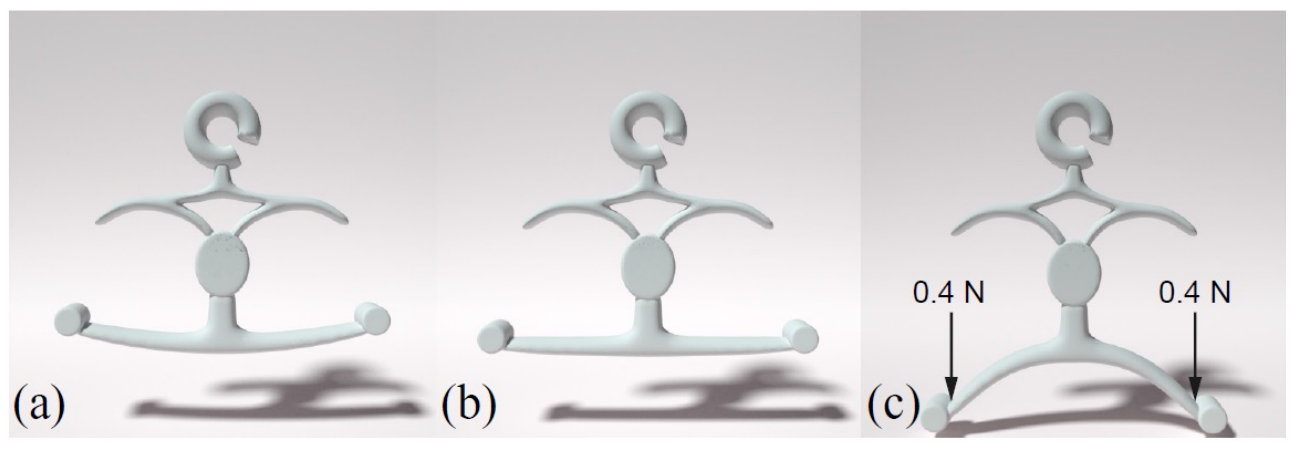
- Interactive force adjustment
- Forward static solver
- Multi-target inverse design



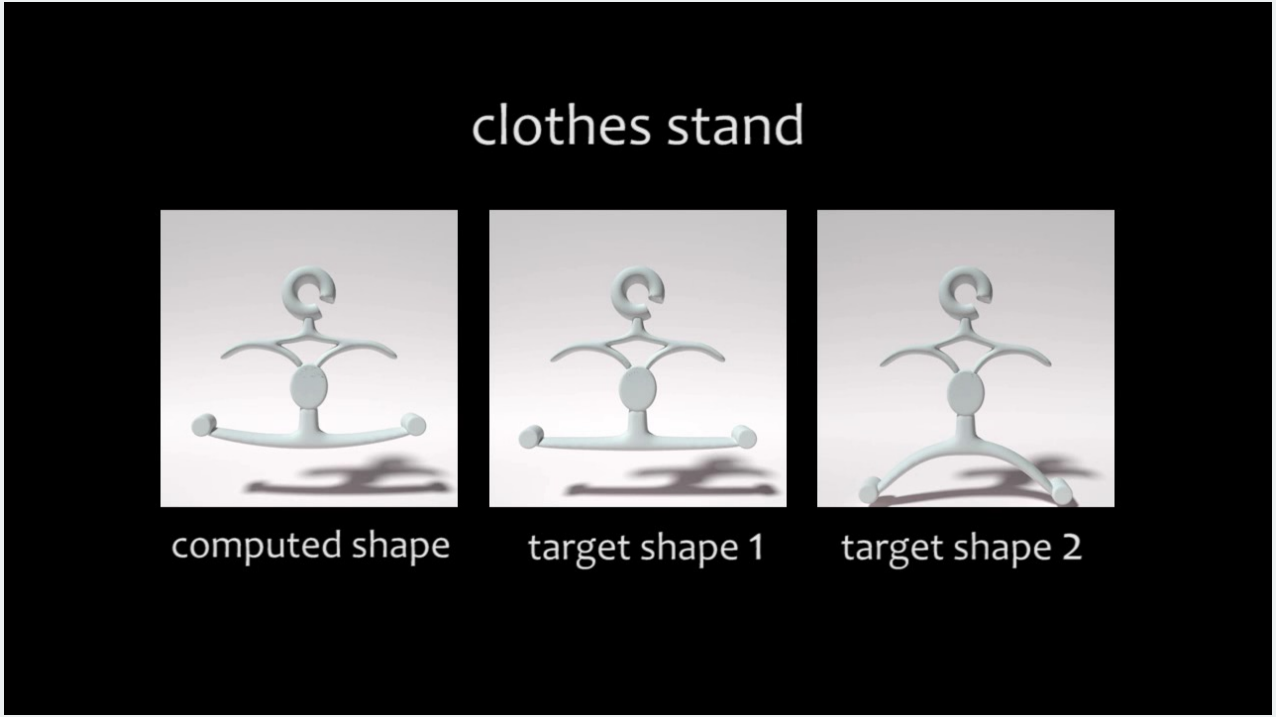
$$f(x_1, X) + g_1 = 0$$
$$f(x_2, X) + g_2 = 0$$
...

Fabrication Results

- Facial hanger

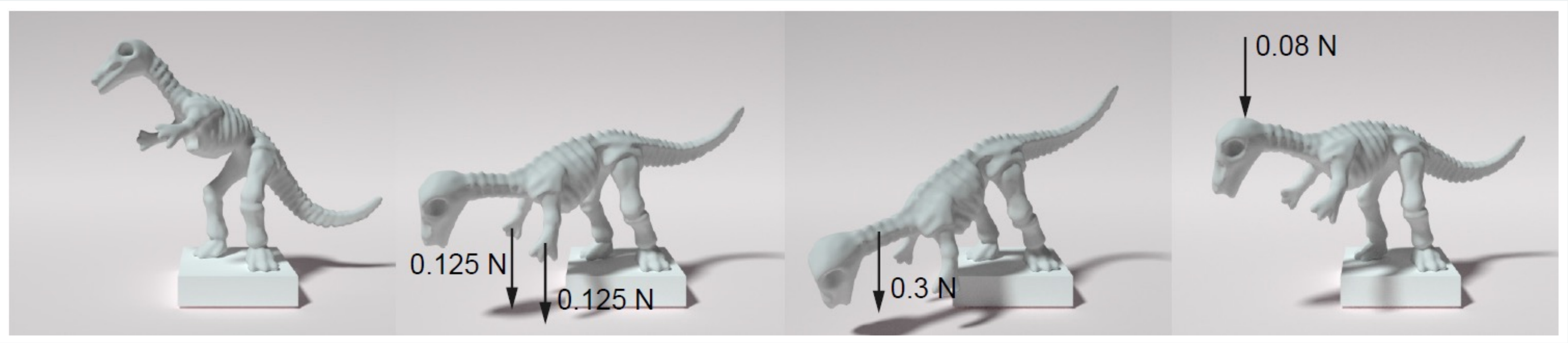


Fabrication Results

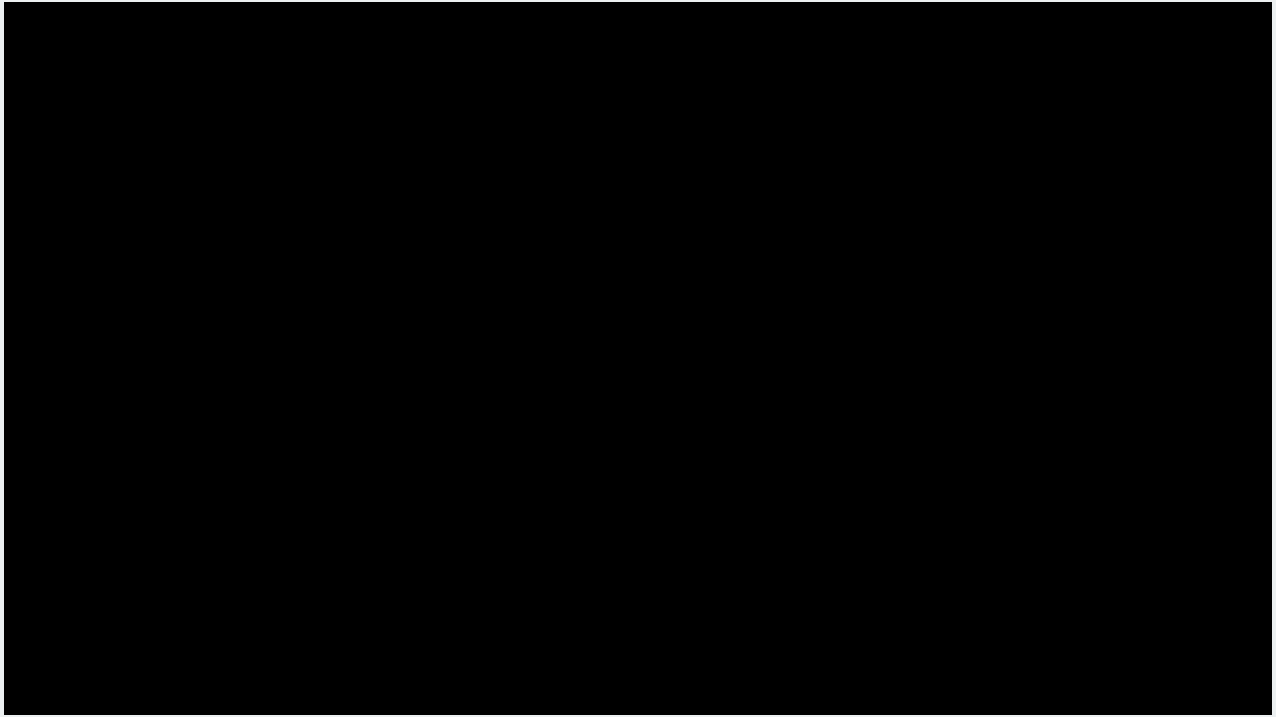


Fabrication Results

- Dinosaur

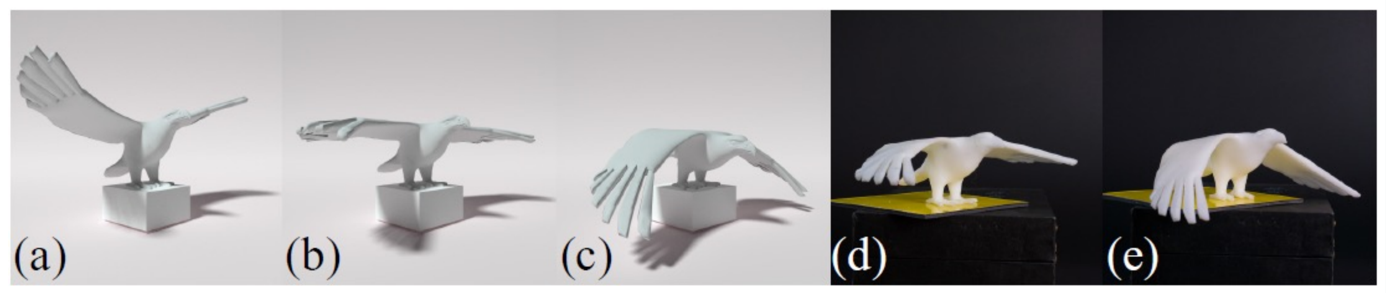


Fabrication Results



Limitations

- Failure case



- No feasible solution for arbitrary input
- Manual mathematical derivation

Conclusion & Future work

- An asymptotic numerical method
- An inverse design tool
- ANM extensions

- Further applications
  - anisotropy
  - partial target
  - various materials

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Timothy Sun

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- Lenovo's Program for Young Scientists

Thank you

Code & Data available at:

http://gaps-zju.org/ANMdesign